

STABILITY OF NETS OF QUADRICS IN \mathbb{P}^5 AND ASSOCIATED DISCRIMINANTS

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ABSTRACT. Let S be a complete intersection surface defined by a net Λ of quadrics in \mathbb{P}^5 . In this paper we analyze GIT stability of nets of quadrics in \mathbb{P}^5 up to projective equivalence, and discuss some connections between a net of quadrics and the associated discriminant sextic curve. In particular, we prove that if S is normal and the discriminant $\Delta(S)$ of S is stable then Λ is stable. And we prove that if S has the reduced discriminant and $\Delta(S)$ is stable then Λ is stable. Moreover, we prove that if S has simple singularities then $\Delta(S)$ has simple singularities.

1. INTRODUCTION

One of central theme in algebraic geometry is to construct compact moduli spaces with geometric meaning. There are two early successes of the moduli theory - the construction and compactification of the moduli spaces of curves $\overline{\mathcal{M}}_g$ and principally polarized abelian varieties (ppavs) $\overline{\mathcal{A}}_g$. While very few other examples are so well understood.

One of the oldest approach to moduli problems is Geometric Invariant Theory (GIT). The GIT stability analysis for pencils of quadrics appear in [3],[2] and [12], it is shown that a pencil of quadrics in \mathbb{P}^n is stable (resp. semistable) if and only if the associated discriminant binary $(n+1)$ -form is non-zero and is GIT stable (resp. semistable) with respect to the natural $SL(2)$ -action. So the GIT stability of a pencil of quadrics can be read off the GIT stability of the associated discriminant locus. But the GIT analysis for nets of quadrics turns out to be more involved. In particular, as Example 3.1 shows, there is no natural correspondence between $SL(6)$ -stability of a net and $SL(3)$ -stability of the associated discriminant curve. Moreover, the complete analysis of stable locus is complicated. For example, see [7] for discussion of GIT stability of net of quadrics in \mathbb{P}^4 . However, we know the following facts: if a net defines a complete intersection surface with simple singularities then the net is stable with respect to $SL(6)$ -action ([18],[16],[17] and [11]). And if a net defines a smooth complete intersection surface, then the associated discriminant curve is stable with respect to $SL(3)$ -action [4].

Firstly, we find GIT stability criterion of net Λ of quadrics in \mathbb{P}^5 via studying special one-parameter subgroups. Our GIT analysis follows the method in Section 2 of [7].

Theorem 1.1. (=Theorem 2.1) *Suppose that Λ is stable with respect to every one-parameter subgroup of the following numerical types:*

- | | |
|--|---|
| (1) $\rho_1 = (1, 1, 1, 1, 1, -5)$. | (8) $\rho_8 = (4, 1, 1, -2, -2, -2)$. |
| (2) $\rho_2 = (1, 1, 1, 1, -2, -2)$. | (9) $\rho_9 = (3, 1, 1, -1, -1, -3)$. |
| (3) $\rho_3 = (1, 1, 1, -1, -1, -1)$. | (10) $\rho_{10} = (2, 1, 0, 0, -1, -2)$. |
| (4) $\rho_4 = (2, 2, -1, -1, -1, -1)$. | (11) $\rho_{11} = (5, 5, -1, -1, -1, -7)$. |
| (5) $\rho_5 = (5, -1, -1, -1, -1, -1)$. | (12) $\rho_{12} = (1, 1, 0, 0, -1, -1)$. |
| (6) $\rho_6 = (2, 2, 2, -1, -1, -4)$. | (13) $\rho_{13} = (5, 3, 1, -1, -3, -5)$. |
| (7) $\rho_7 = (7, 1, 1, 1, -5, -5)$. | |

Then Λ is stable.

On the basis of this partial analysis, we may already conclude the important fact that a stable net has a pure two-dimensional intersection, and hence defines a connected surface with local complete intersection singularities.

Corollary 1.2. (=Theorem 2.5) *If a net of quadrics in \mathbb{P}^5 is stable, then the corresponding intersection is connected and purely two-dimensional.*

And our analysis makes us possible to discuss some connections between a net of quadrics and the associated discriminant sextic curve.

More precisely, we prove the following theorems.

Theorem 1.3. (=Theorem 3.3) *Suppose a net Λ of quadrics in \mathbb{P}^5 defines a complete intersection normal surface S . If the discriminant $\Delta(S)$ of S is stable then Λ is stable.*

Theorem 1.4. (=Theorem 3.4) *Suppose a net Λ of quadrics in \mathbb{P}^5 defines a complete intersection surface S with the reduced discriminant $\Delta(S)$. If $\Delta(S)$ is stable then Λ is stable.*

Let S be a K3 surface of degree 8 in \mathbb{P}^5 , given by the complete intersection of three quadrics. Associated to S is a K3 surface S' which is a double cover of \mathbb{P}^2 ramified over a sextic. And there is a dominant rational map $\phi : \mathcal{N}^{ss} // SL(6) \dashrightarrow \mathcal{C}_6^{ss} // SL(3)$ where \mathcal{N} the space of nets of quadrics in \mathbb{P}^5 and \mathcal{C}_6 the space of plane sextic curves. The degree of this map is equal to the number of non-effective theta characteristics on a general sextic curve which is $2^9(2^{10} + 1)$ (cf. [6] and [19]). Our original motivation in this paper is to compare the moduli space of algebraic K3 surfaces with degree 8 with the moduli space of K3 surfaces as a double cover of \mathbb{P}^2 , ramified over a sextic curve. It is well known that if S is nonsingular then $\Delta(S)$ is stable [4].

Theorem 1.5. (=Theorem 3.5) *Suppose a net Λ of quadrics in \mathbb{P}^5 defines a complete intersection surface S . If S has simple singularities, then $\Delta(S)$ has simple singularities.*

The morphism from the moduli space of degree 8 K3 surfaces to degree 2 K3 surfaces has been studied from many points of view, starting from Mukai's paper [13]. His result implies directly that if S is a smooth K3 surface which is a complete intersection of three quadrics in \mathbb{P}^5 then $\Delta(S)$ has simple singularities.

It has an interpretation in Hodge-theoretic terms that should yield Theorem 1.5 fairly quickly. One can compare the Hodge structures on the two K3 surfaces and interprets what it means for their images to have simple singularities in terms of (-2) -class in the Picard group [10]. Our approach is rather direct via GIT analysis even though it involves complicated computations.

In Section 2, we describe a finite set of numerical types of one parameter subgroups $\{\rho_i\}_{i=1}^N$ such that the union of the ρ_i -nonstable points is $Gr(3, W) \setminus Gr(3, W)^s$. In Section 3, we use this result to prove our main theorems. In this paper, we work on the field of complex numbers.

2. STABILITY OF NETS OF QUADRICS

Our notations and GIT analysis follow Section 2 of [7].

Let $V = H^0(\mathbb{P}^5, \mathcal{O}(1))$ and let $W = H^0(\mathbb{P}^5, \mathcal{O}(2))$ be the space of quadratic forms. A net of quadrics in \mathbb{P}^5 is by definition a plane in $\mathbb{P}(\bigwedge^3 W)$. So the space of nets of quadrics is by definition the Grassmannian $Gr(3, W) \subset \mathbb{P}(\bigwedge^3 W)$.

Let $\rho = (a_0, a_1, a_2, a_3, a_4, a_5) : \mathbb{C}^* \rightarrow SL(6)$ be a normalized one-parameter subgroup (1-PS), i.e. ρ is a one-parameter subgroup, acting diagonally on a basis $\{x_0, x_1, \dots, x_5\}$ of V with weights $\{a_0, a_1, a_2, a_3, a_4, a_5\}$ satisfying $a_0 \geq a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5$ and $\sum_{k=0}^5 a_k = 0$.

Then the ρ -weight of a quadratic monomial $x_i x_j$ is

$$\omega_\rho(x_i x_j) = a_i + a_j$$

and the ρ -weight of the Plücker coordinate $x_{i_1} x_{j_1} \wedge x_{i_2} x_{j_2} \wedge x_{i_3} x_{j_3}$ is simply $\sum_{k=1}^3 \omega_\rho(x_{i_k} x_{j_k})$.

By the Hilbert-Mumford numerical criterion (Theorem 2.1 in [15]), a net Λ is stable (resp., semistable) with respect to ρ if there exists a Plücker coordinate that does not vanish on Λ with positive (resp., non-negative) ρ -weight. And Λ is stable (resp., semistable) if and only if Λ is stable (resp., semistable) with respect to all one-parameter subgroups.

A priori, the numerical criterion requires one to check ρ -stability for all one-parameter subgroups. However, there necessarily exists a finite set of numerical types of one-parameter subgroups $\{\rho_i\}_{i=1}^N$ such that the union of the ρ_i -nonstable points is $Gr(3, W) \setminus Gr(3, W)^s$. The first main result of this section, Theorem 2.1, describes such a set of one-parameter subgroups explicitly.

Throughout this section, we use the following notations. Given a basis $\{x_0, x_1, \dots, x_5\}$ of V and a normalized 1-PS ρ acting on $\{x_0, x_1, \dots, x_5\}$, we can define two complete orderings on the set of quadratic monomials:

- (1) The lexicographic ordering " \succ_{lex} ",
- (2) " \succ_ρ ": $m_1 \succ_\rho m_2$ if either $\omega_\rho(m_1) > \omega_\rho(m_2)$ or $\omega_\rho(m_1) = \omega_\rho(m_2)$ and $m_1 \succ_{lex} m_2$.

And there is another ordering " \geq ", according to which $m_1 \geq m_2$ if and only if $\omega_\rho(m_1) \geq \omega_\rho(m_2)$ for any normalized 1-PS acting diagonally on $\{x_0, x_1, \dots, x_5\}$. Note that $m_1 \geq m_2$ implies $m_1 \succeq_{lex} m_2$.

For any quadric $Q \in W$, we denote by $in_{lex}(Q)$ the initial monomial of Q with respect to \succ_{lex} and if ρ is normalized 1-PS acting on $\{x_0, x_1, \dots, x_5\}$, we denote $in_\rho(Q)$ by the initial monomial of Q with respect to \succ_ρ .

For any net $\Lambda = (Q_1, Q_2, Q_3)$, we can always choose a basis (Q'_1, Q'_2, Q'_3) with $\text{in}_{lex}(Q'_1) \succ_{lex} \text{in}_{lex}(Q'_2) \succ_{lex} \text{in}_{lex}(Q'_3)$ by replacing Q_1, Q_2, Q_3 with a linear combination of the three polynomials. We call such a basis (Q'_1, Q'_2, Q'_3) normalized basis of Λ .

Finally, given a basis $\{x_0, x_1, \dots, x_5\}$ of V , we define the distinguished flag $F_0 \subset F_1 \subset F_2 \subset F_3 \subset F_4 \subset \mathbb{P}(V)$ as follows:

$$F_0 : x_1 = x_2 = x_3 = x_4 = x_5 = 0,$$

$$F_1 : x_2 = x_3 = x_4 = x_5 = 0,$$

$$F_2 : x_3 = x_4 = x_5 = 0,$$

$$F_3 : x_4 = x_5 = 0,$$

$$F_4 : x_5 = 0.$$

Theorem 2.1. *Suppose that Λ is stable with respect to every one-parameter subgroup of the following numerical types:*

- | | |
|--|---|
| (1) $\rho_1 = (1, 1, 1, 1, 1, -5)$. | (8) $\rho_8 = (4, 1, 1, -2, -2, -2)$. |
| (2) $\rho_2 = (1, 1, 1, 1, -2, -2)$. | (9) $\rho_9 = (3, 1, 1, -1, -1, -3)$. |
| (3) $\rho_3 = (1, 1, 1, -1, -1, -1)$. | (10) $\rho_{10} = (2, 1, 0, 0, -1, -2)$. |
| (4) $\rho_4 = (2, 2, -1, -1, -1, -1)$. | (11) $\rho_{11} = (5, 5, -1, -1, -1, -7)$. |
| (5) $\rho_5 = (5, -1, -1, -1, -1, -1)$. | (12) $\rho_{12} = (1, 1, 0, 0, -1, -1)$. |
| (6) $\rho_6 = (2, 2, 2, -1, -1, -4)$. | (13) $\rho_{13} = (5, 3, 1, -1, -3, -5)$. |
| (7) $\rho_7 = (7, 1, 1, 1, -5, -5)$. | |

Then Λ is stable.

Remark 2.2. *By the above Theorem, Λ is stable with respect to a fixed torus T if and only if it is stable with respect to all one-parameter subgroups in T of the numerical types $\{\rho_i\}_{i=1}^{13}$.*

Fix a net Λ which is ρ_i -stable for each $1 \leq i \leq 13$. By the Hilbert-Mumford numerical criterion, to prove that Λ is stable, we must show that Λ is stable with respect to an arbitrary 1-PS $\chi : \mathbb{C}^* \rightarrow SL(6)$. Without loss of generality, we may assume that χ is normalized, acting diagonally on the basis $\{x_0, \dots, x_5\}$ with weights (a, b, c, d, e, f) . To prove the theorem, we must exhibit a Plücker coordinates that does not vanish on Λ with positive χ -weight. More explicitly, if (Q_1, Q_2, Q_3) is a normalized basis of Λ , we must exhibit non-zero quadratic monomials m_1, m_2, m_3 in the variables $\{x_0, \dots, x_5\}$ which appear with non-zero coefficient in $Q_1 \wedge Q_2 \wedge Q_3$ with $\sum_{k=1}^3 \omega_\chi(m_k) > 0$. We begin with a preparatory lemma.

Lemma 2.3. *If a net Λ is ρ_i -stable for each $1 \leq i \leq 5$, then for a basis $\{x_0, \dots, x_5\}$ of V , the normalized basis (Q_1, Q_2, Q_3) of Λ satisfies the following:*

- (1) $Q_1, Q_2, Q_3 \notin (x_5)$.
- (2) $(Q_2, Q_3) \not\subseteq (x_4, x_5)$ and $Q_3 \notin (x_4, x_5)^2$.
- (3) $(Q_1, Q_2, Q_3) \not\subseteq (x_3, x_4, x_5)$ and either $(Q_2, Q_3) \not\subseteq (x_3, x_4, x_5)$ or $Q_3 \notin (x_3, x_4, x_5)^2$.
- (4) $(Q_2, Q_3) \not\subseteq (x_2, x_3, x_4, x_5)^2$ and either $(Q_1, Q_2, Q_3) \not\subseteq (x_2, x_3, x_4, x_5)$ or $Q_3 \notin (x_2, x_3, x_4, x_5)^2$.

- (5) $(Q_1, Q_2, Q_3) \not\subseteq (x_1, x_2, x_3, x_4, x_5)$ or $(Q_2, Q_3) \not\subseteq (x_1, x_2, x_3, x_4, x_5)^2$.
(i.e. $\text{in}_{\text{lex}}(Q_1) = x_0^2$ or $\text{in}_{\text{lex}}(Q_1), \text{in}_{\text{lex}}(Q_2) \in (x_0)$.)

Proof. (1), (2), (3), (4), (5) follows immediately from $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5$ -stability of Λ , respectively. \square

Let $M_i = \text{in}_{\text{lex}}(Q_i)$ for $i = 1, 2, 3$. We can now begin the proof of Theorem 2.1.

Proof of Theorem 2.1. We consider separately the following three cases:

- I. F_0 is not in the base locus of Λ ;
- II. F_0 is in the base locus of Λ but F_1 is not;
- III. F_1 is in the base locus of Λ .

⊙Case I: F_0 is not a base point. We have $M_1 = x_0^2$.

•Case I.1: Q_2 has a term $m_2 \geq x_2^2$. Then Q_3 has a term $m_3 \geq x_4^2$, $M_3 \geq x_3x_5$ and $M_2 \geq x_1x_5$ by Lemma 0.1(1),(2) and (4), respectively. Suppose Λ is not χ -stable, then

$$\omega_\chi(M_1) + \begin{pmatrix} \omega_\chi(M_2) \\ \omega_\chi(m_2) \end{pmatrix} + \begin{pmatrix} \omega_\chi(M_3) \\ \omega_\chi(m_3) \end{pmatrix} \leq 0 \iff 2a + \begin{pmatrix} b+f \\ 2c \end{pmatrix} + \begin{pmatrix} d+f \\ 2e \end{pmatrix} \leq 0.$$

Then $a = b = c = d = e = f = 0$. So we get a contradiction.

•Case I.2: Q_2 has no term $\geq x_2^2$ (i.e. $M_2 < x_2^2$). Then Q_3 has a term $m_3 \geq x_4^2$, Q_2 has a term $m_2 \geq x_3^2$, $M_3 \geq x_2x_5$ and $M_2 \geq x_1x_5$ by Lemma 0.1(1),(2),(3) and (4), respectively. Suppose Λ is not χ -stable, then

$$\omega_\chi(M_1) + \begin{pmatrix} \omega_\chi(M_2) \\ \omega_\chi(m_2) \end{pmatrix} + \begin{pmatrix} \omega_\chi(M_3) \\ \omega_\chi(m_3) \end{pmatrix} \leq 0 \iff 2a + \begin{pmatrix} b+f \\ 2d \end{pmatrix} + \begin{pmatrix} c+f \\ 2e \end{pmatrix} \leq 0.$$

Then $\chi = (2k, 2k, 2k, -k, -k, -4k)$. This contradicts to ρ_6 -stability.

⊙Case II: F_0 is a base point but F_1 is not in the base locus. By Lemma 0.1, we have the following conditions:

- (1) Q_3 has a term $\geq x_4^2$.
- (2) Q_2 has a term $\geq x_3^2$ and $M_3 \geq x_3x_5$.
- (3) Either Q_2 has a term $\geq x_2^2$ or $M_3 \geq x_2x_5$.
- (4) Q_1 has a term $\geq x_1^2$.
- (5) $M_1, M_2 \in \{x_0x_1, x_0x_2, x_0x_3, x_0x_4, x_0x_5\}$.

•Case II.1: Q_2 contains a term $m_2 \geq x_2^2$. Then

- (1) Q_3 has a term $\geq x_4^2$.
- (2) $M_3 \geq x_3x_5$.
- (3) Q_2 has a term $\geq x_2^2$.
- (4) Q_1 has a term $\geq x_1^2$.
- (5) $M_1, M_2 \in \{x_0x_1, x_0x_2, x_0x_3, x_0x_4, x_0x_5\}$.

•Case II.1.a: $M_1 = x_0x_4$ and $M_2 = x_0x_5$.

By $\rho_7 = (7, 1, 1, 1, -5, -5)$ -stability, Q_3 has a term $\geq x_3^2$. Suppose Λ is not χ -stable, then

$$\binom{a+e}{2b} + \binom{a+f}{2c} + 2d \leq 0.$$

Then $\chi = (4k, k, k, -2k, -2k, -2k)$. This contradicts to ρ_8 -stability.

Case II.1.b: $M_1 = x_0x_3$ and $M_2 = x_0x_5$.

By $\rho_8 = (4, 1, 1, -2, -2, -2)$ -stability, Q_3 has a term $\geq x_2x_5$. Suppose Λ is not χ -stable, then

$$\binom{a+d}{2b} + \binom{a+f}{2c} + \binom{c+f}{2e} \leq 0$$

Then $\chi = (2k, k, 0, 0, -k, -2k)$. This contradicts to ρ_{10} -stability.

Case II.1.c: $M_1 = x_0x_3$ and $M_2 = x_0x_4$.

By $\rho_8 = (4, 1, 1, -2, -2, -2)$ -stability, Q_3 has a term $\geq x_2x_5$. Suppose Λ is not χ -stable, then

$$\binom{a+d}{2b} + \binom{a+e}{2c} + \binom{c+f}{2e} \leq 0$$

Then $a = b = c = d = e = f = 0$. So we get a contradiction.

Case II.1.d: $M_1 = x_0x_2$ and $M_2 = x_0x_5$.

Suppose Λ is not χ -stable, then

$$\binom{a+c}{2b} + \binom{a+f}{2c} + \binom{d+f}{2e} \leq 0$$

Then $\chi = (2k, k, 0, 0, -k, -2k)$. This contradicts to ρ_{10} -stability.

Case II.1.e: $M_1 = x_0x_2$ and $M_2 = x_0x_4$.

Suppose Λ is not χ -stable, then

$$\binom{a+c}{2b} + \binom{a+e}{2c} + \binom{d+f}{2e} \leq 0$$

Then $\chi = (k, k, k, k, -2k, -2k)$. This contradicts to ρ_2 -stability.

Case II.1.f: $M_1 = x_0x_2$ and $M_2 = x_0x_3$.

Suppose Λ is not χ -stable, then

$$\binom{a+c}{2b} + \binom{a+d}{2c} + \binom{d+f}{2e} \leq 0$$

Then $a = b = c = d = e = f = 0$. We get a contradiction.

Case II.1.g: $M_1 = x_0x_1$ and $M_2 = x_0x_5$.

Suppose Λ is not χ -stable, then

$$(a+b) + \binom{a+f}{2c} + \binom{d+f}{2e} \leq 0$$

Then $a = b = c = d = e = f = 0$. We get a contradiction.

•**Case II.1.h:** $M_1 = x_0x_1$ and $M_2 = x_0x_4$.

Suppose Λ is not χ -stable, then

$$(a+b) + \binom{a+e}{2c} + \binom{d+f}{2e} \leq 0$$

Then $a = b = c = d = e = f = 0$. We get a contradiction.

•**Case II.1.i:** $M_1 = x_0x_1$ and $M_2 = x_0x_3$.

Suppose Λ is not χ -stable, then

$$(a+b) + \binom{a+d}{2c} + \binom{d+f}{2e} \leq 0$$

Then $a = b = c = d = e = f = 0$. We get a contradiction.

•**Case II.1.j:** $M_1 = x_0x_1$ and $M_2 = x_0x_2$.

Suppose Λ is not χ -stable, then

$$(a+b) + (a+c) + \binom{d+f}{2e} \leq 0$$

Then $a = b = c = d = e = f = 0$. We get a contradiction.

•**Case II.2:** Q_2 has no term $m_2 \geq x_2^2$ (i.e. $(Q_2, Q_3) \subset (x_3, x_4, x_5)$). Then

- (1) Q_3 has a term $\geq x_4^2$.
- (2) Q_2 has a term $\geq x_3^2$.
- (3) $M_3 \geq x_2x_5$.
- (4) Q_1 has a term $\geq x_1^2$.
- (5) $M_1, M_2 \in \{x_0x_1, x_0x_2, x_0x_3, x_0x_4, x_0x_5\}$.

•**Case II.2.a:** $M_1 = x_0x_4$ and $M_2 = x_0x_5$.

By $\rho_7 = (7, 1, 1, 1, -5, -5)$ -stability, Q_3 has a term $\geq x_3^2$. And by $\rho_9 = (3, 1, 1, -1, -1, -3)$ -stability, Q_3 has a term $\geq x_2x_4$. Suppose Λ is not χ -stable, then

$$\binom{a+e}{2b} + \binom{a+f}{2d} + \binom{c+e}{2d} \leq 0$$

Then $a = b = c = d = e = f = 0$. We get a contradiction.

•**Case II.2.b:** $M_1 = x_0x_3$ and $M_2 = x_0x_5$.

By $\rho_9 = (3, 1, 1, -1, -1, -3)$ -stability, Q_3 has a term $\geq x_2x_4$. Suppose Λ is not χ -stable, then

$$\binom{a+d}{2b} + \binom{a+f}{2d} + (c+e) \leq 0$$

Then $a = b = c = d = e = f = 0$. We get a contradiction.

Case II.2.c: $M_1 = x_0x_3$ and $M_2 = x_0x_4$.

Suppose Λ is not χ -stable, then

$$\binom{a+d}{2b} + \binom{a+e}{2d} + \binom{c+f}{2e} \leq 0$$

Then $a = b = c = d = e = f = 0$. We get a contradiction.

Case II.2.d: $M_1 = x_0x_2$ and $M_2 = x_0x_5$.

If Q_2 has a term $\geq x_1x_3$, then Λ is χ -stable. Indeed, if Λ is not χ -stable, then

$$\binom{a+c}{2b} + \binom{a+f}{b+d} + \binom{c+f}{2e} \leq 0$$

Then $a = b = c = d = e = f = 0$. We get a contradiction.

Now assume Q_2 has no term $\geq x_1x_3$. Then by $\rho_{10} = (2, 1, 0, 0, -1, -2)$ -stability, Q_3 has a term $\geq x_1x_5$ or $\geq x_3x_4$. If Q_3 has a term $\geq x_1x_5$, then Λ is χ -stable. Indeed, if Λ is not χ -stable, then

$$\binom{a+c}{2b} + \binom{a+f}{2d} + \binom{b+f}{2e} \leq 0$$

Then $\chi = (2k, 2k, 2k, -k, -k, -4k)$. This contradicts to ρ_6 -stability.

If Q_3 has a term $\geq x_3x_4$, then Λ is χ -stable. Indeed, if Λ is not χ -stable, then

$$\binom{a+c}{2b} + \binom{a+f}{2d} + \binom{c+f}{d+e} \leq 0$$

Then $\chi = (2k, 2k, 2k, -k, -k, -4k)$. This contradicts to ρ_6 -stability.

Case II.2.e: $M_1 = x_0x_2$ and $M_2 = x_0x_4$.

Suppose Λ is not χ -stable, then

$$\binom{a+c}{2b} + \binom{a+e}{2d} + \binom{c+f}{2e} \leq 0$$

Then $a = b = c = d = e = f = 0$. We get a contradiction.

Case II.2.f: $M_1 = x_0x_2$ and $M_2 = x_0x_3$.

Suppose Λ is not χ -stable, then

$$\binom{a+c}{2b} + (a+d) + \binom{c+f}{2e} \leq 0$$

Then $a = b = c = d = e = f = 0$. We get a contradiction.

•**Case II.2.g:** $M_1 = x_0x_1$ and $M_2 = x_0x_5$.

Suppose Λ is not χ -stable, then

$$(a+b) + \binom{a+f}{2d} + \binom{c+f}{2e} \leq 0$$

Then $\chi = (2k, 2k, 2k, -k, -k, -4k)$. This contradicts to ρ_6 -stability.

•**Case II.2.h:** $M_1 = x_0x_1$ and $M_2 = x_0x_4$.

Suppose Λ is not χ -stable, then

$$(a+b) + \binom{a+e}{2d} + \binom{c+f}{2e} \leq 0$$

Then $a = b = c = d = e = f = 0$. We get a contradiction.

•**Case II.2.i:** $M_1 = x_0x_1$ and $M_2 = x_0x_3$.

Suppose Λ is not χ -stable, then

$$(a+b) + (a+d) + \binom{c+f}{2e} \leq 0$$

Then $a = b = c = d = e = f = 0$. We get a contradiction.

⊙**Case III: F_1 is in the base locus.** By Lemma 0.1, we have the following conditions:

- (1) Q_3 has a term $\geq x_4^2$.
- (2) Q_2 has a term $\geq x_3^2$.
- (3) Q_1 has a term $\geq x_2^2$ and either Q_2 has a term $\geq x_2^2$ or $M_3 \geq x_2x_5$.
- (4) Q_1 has no term $\geq x_1^2$ (i.e. $Q_1, Q_2, Q_3 \in (x_2, x_3, x_4, x_5)$) and $M_3 \geq x_1x_5$.
- (5) $M_1, M_2 \in \{x_0x_2, x_0x_3, x_0x_4, x_0x_5\}$.

•**Case III.1:** Q_2 has a term $\geq x_2^2$. Then

- (1) Q_3 has a term $\geq x_4^2$.
- (2) \times
- (3) Q_1 has a term $\geq x_2^2$ and Q_2 has a term $\geq x_2^2$.
- (4) $Q_1, Q_2, Q_3 \in (x_2, x_3, x_4, x_5)$ and $M_3 \geq x_1x_5$.
- (5) $M_1, M_2 \in \{x_0x_2, x_0x_3, x_0x_4, x_0x_5\}$.

•**Case III.1.a:** $M_1 = x_0x_4$ and $M_2 = x_0x_5$.

By $\rho_7 = (7, 1, 1, 1, -5, -5)$ -stability, Q_3 has a term $\geq x_3^2$. If Q_2 has a term $\geq x_1x_4$, then Λ is χ -stable. Indeed, if Λ is not χ -stable, then

$$\binom{a+e}{2c} + \binom{a+f}{2c} + \binom{b+f}{2d} \leq 0$$

Then $\chi = (k, k, 0, 0, -k, -k)$. This contradicts to ρ_{12} -stability.

Now assume Q_2 has no term $\geq x_1x_4$, then by $\rho_{11} = (5, 5, -1, -1, -1, -7)$ -stability, Q_3 has a term $\geq x_1x_4$. Suppose Λ is not χ -stable, then

$$\begin{pmatrix} a+e \\ 2c \end{pmatrix} + \begin{pmatrix} a+f \\ 2c \end{pmatrix} + \begin{pmatrix} b+e \\ 2d \end{pmatrix} \leq 0$$

Then $\chi = (k, k, 0, 0, -k, -k)$. This contradicts to ρ_{12} -stability.

•Case III.1.b: $M_1 = x_0x_3$ and $M_2 = x_0x_5$.

Suppose Λ is not χ -stable, then

$$\begin{pmatrix} a+d \\ 2c \end{pmatrix} + \begin{pmatrix} a+f \\ 2c \end{pmatrix} + \begin{pmatrix} b+f \\ 2e \end{pmatrix} \leq 0$$

Then $\chi = (5k, 5k, -k, -k, -k, -7k)$. This contradicts to ρ_{11} -stability.

•Case III.1.c: $M_1 = x_0x_3$ and $M_2 = x_0x_4$.

Suppose Λ is not χ -stable, then

$$\begin{pmatrix} a+d \\ 2c \end{pmatrix} + \begin{pmatrix} a+e \\ 2c \end{pmatrix} + \begin{pmatrix} b+f \\ 2e \end{pmatrix} \leq 0$$

Then $a = b = c = d = e = f = 0$. We get a contradiction.

•Case III.1.d: $M_1 = x_0x_2$ and $M_2 = x_0x_5$.

Suppose Λ is not χ -stable, then

$$(a+c) + \begin{pmatrix} a+f \\ 2c \end{pmatrix} + \begin{pmatrix} b+f \\ 2e \end{pmatrix} \leq 0$$

Then $\chi = (5k, 5k, -k, -k, -k, -7k)$. This contradicts to ρ_{11} -stability.

•Case III.1.e: $M_1 = x_0x_2$ and $M_2 = x_0x_4$.

Suppose Λ is not χ -stable, then

$$(a+c) + \begin{pmatrix} a+e \\ 2c \end{pmatrix} + \begin{pmatrix} b+f \\ 2e \end{pmatrix} \leq 0$$

Then $a = b = c = d = e = f = 0$. We get a contradiction.

•Case III.1.f: $M_1 = x_0x_2$ and $M_2 = x_0x_3$.

Suppose Λ is not χ -stable, then

$$(a+c) + \begin{pmatrix} a+d \\ 2c \end{pmatrix} + \begin{pmatrix} b+f \\ 2e \end{pmatrix} \leq 0$$

Then $a = b = c = d = e = f = 0$. We get a contradiction.

•Case III.2: Q_2 has no term $\geq x_2^2$. (i.e. $(Q_2, Q_3) \subset (x_3, x_4, x_5)$). Then

- (1) Q_3 has a term $\geq x_4^2$.
- (2) Q_2 has a term $\geq x_3^2$.
- (3) Q_1 has a term $\geq x_2^2$.
- (4) $Q_1, Q_2, Q_3 \in (x_2, x_3, x_4, x_5)$ and $M_3 \geq x_1x_5$.
- (5) $M_1, M_2 \in \{x_0x_2, x_0x_3, x_0x_4, x_0x_5\}$.

Case III.2.a: $M_1 = x_0x_4$ and $M_2 = x_0x_5$.

By $\rho_7 = (7, 1, 1, 1, -5, -5)$ -stability, Q_3 has a term $\geq x_3^2$. And by $\rho_9 = (3, 1, 1, -1, -1, -3)$ -stability, Q_3 has a term $\geq x_2x_4$. If Q_1 has a term $\geq x_1x_2$, then Λ is χ -stable. Indeed, if Λ is not χ -stable, then

$$\begin{pmatrix} a+e \\ b+c \end{pmatrix} + \begin{pmatrix} a+f \\ 2d \end{pmatrix} + \begin{pmatrix} b+f \\ c+e \\ 2d \end{pmatrix} \leq 0$$

Then $\chi = (5k, 5k, -k, -k, -k, -7k)$. This contradicts to ρ_{11} -stability.

Now assume that Q_1 has no term $\geq x_1x_2$. If Q_3 has a term $\geq x_1x_4$, then Λ is χ -stable. Indeed, if Λ is not χ -stable, then

$$\begin{pmatrix} a+e \\ 2c \end{pmatrix} + \begin{pmatrix} a+f \\ 2d \end{pmatrix} + \begin{pmatrix} b+e \\ 2d \end{pmatrix} \leq 0$$

Then $\chi = (k, k, 0, 0, -k, -k)$. This contradicts to ρ_{12} -stability.

If Q_3 has no term $\geq x_1x_4$, then by $\rho_{11} = (5, 5, -1, -1, -1, -7)$ -stability, Q_2 has a term $\geq x_1x_4$.

If Q_3 also has a term $\geq x_2x_3$. Then Λ is χ -stable. Indeed, if Λ is not χ -stable, then

$$\begin{pmatrix} a+e \\ 2c \end{pmatrix} + \begin{pmatrix} a+f \\ 2d \\ b+e \end{pmatrix} + \begin{pmatrix} b+f \\ c+d \end{pmatrix} \leq 0$$

Then $\chi = (k, k, 0, 0, -k, -k)$. This contradicts to ρ_{12} -stability.

Now assume Q_3 has no term $\geq x_2x_3$. Then by $\rho_{13} = (5, 3, 1, -1, -3, -5)$ -stability, Q_2 has a term $\geq x_1x_3$. Suppose Λ is not χ -stable, then

$$\begin{pmatrix} a+e \\ 2c \end{pmatrix} + \begin{pmatrix} a+f \\ b+d \end{pmatrix} + \begin{pmatrix} b+f \\ 2d \\ c+e \end{pmatrix} \leq 0$$

Then $a = b = c = d = e = f = 0$. We get a contradiction.

Case III.2.b: $M_1 = x_0x_3$ and $M_2 = x_0x_5$.

By $\rho_9 = (3, 1, 1, -1, -1, -3)$ -stability, Q_3 has a term $\geq x_2x_4$. If Q_1 has a term $\geq x_1x_2$, then Λ is χ -stable. Indeed, if Λ is not χ -stable, then

$$\begin{pmatrix} a+d \\ b+c \end{pmatrix} + \begin{pmatrix} a+f \\ 2d \end{pmatrix} + \begin{pmatrix} b+f \\ c+e \end{pmatrix} \leq 0$$

Then $\chi = (5k, 5k, -k, -k, -k, -7k)$. This contradicts to ρ_{11} -stability.

Now assume Q_1 has no term $\geq x_1x_2$. If Q_3 has a term $\geq x_1x_4$, then Λ is χ -stable. Indeed, if Λ is not χ -stable, then

$$\begin{pmatrix} a+d \\ 2c \end{pmatrix} + \begin{pmatrix} a+f \\ 2d \end{pmatrix} + (b+e) \leq 0$$

Then $a = b = c = d = e = f = 0$. We get a contradiction.

Assume now Q_3 has no term $\geq x_1x_4$. By $\rho_{11} = (5, 5, -1, -1, -1, -7)$ -stability, Q_2 has a term $\geq x_1x_4$. Suppose Λ is not χ -stable, then

$$\begin{pmatrix} a+d \\ 2c \end{pmatrix} + \begin{pmatrix} a+f \\ 2d \\ b+e \end{pmatrix} + \begin{pmatrix} b+f \\ c+e \end{pmatrix} \leq 0$$

Then $a = b = c = d = e = f = 0$. We get a contradiction.

Case III.2.c: $M_1 = x_0x_3$ and $M_2 = x_0x_4$.

Suppose Λ is not χ -stable, then

$$\begin{pmatrix} a+d \\ 2c \end{pmatrix} + \begin{pmatrix} a+e \\ 2d \end{pmatrix} + \begin{pmatrix} b+f \\ 2e \end{pmatrix} \leq 0$$

Then $a = b = c = d = e = f = 0$. We get a contradiction.

Case III.2.d: $M_1 = x_0x_2$ and $M_2 = x_0x_5$.

If Q_2 has a term $\geq x_1x_4$. Then Λ is χ -stable. Indeed, if Λ is not χ -stable, then

$$(a+c) + \begin{pmatrix} a+f \\ 2d \\ b+e \end{pmatrix} + \begin{pmatrix} b+f \\ 2e \end{pmatrix} \leq 0$$

Then $a = b = c = d = e = f = 0$. We get a contradiction.

Now assume Q_2 has no term $\geq x_1x_4$. By $\rho_{11} = (5, 5, -1, -1, -1, -7)$ -stability, Q_3 has a term $\geq x_1x_4$. Suppose Λ is not χ -stable, then

$$(a+c) + \begin{pmatrix} a+f \\ 2d \end{pmatrix} + (b+e) \leq 0$$

Then $a = b = c = d = e = f = 0$. We get a contradiction.

Case III.2.e: $M_1 = x_0x_2$ and $M_2 = x_0x_4$.

Suppose Λ is not χ -stable, then

$$(a+c) + \begin{pmatrix} a+e \\ 2d \end{pmatrix} + \begin{pmatrix} b+f \\ 2e \end{pmatrix} \leq 0$$

Then $a = b = c = d = e = f = 0$. We get a contradiction.

Case III.2.f: $M_1 = x_0x_2$ and $M_2 = x_0x_3$.

Suppose Λ is not χ -stable, then

$$(a + c) + (a + d) + \binom{b + f}{2e} \leq 0$$

Then $a = b = c = d = e = f = 0$. We get a contradiction. \square

Lemma 2.4. *A net Λ is not stable with respect to one of $\{\rho_i\}_{i=1}^5$ if and only if it satisfies one of the following conditions with respect to a distinguished flag $F_0 \subset F_1 \subset F_2 \subset F_3 \subset F_4 \subset \mathbb{P}^5$.*

- (1) $\rho_1 = (1, 1, 1, 1, 1, -5)$:
 - (a) An element of Λ contains F_4 .
- (2) $\rho_2 = (1, 1, 1, 1, -2, -2)$:
 - (a) A pencil of Λ contains F_3 , or
 - (b) An element of Λ is singular along F_3 .
- (3) $\rho_3 = (1, 1, 1, -1, -1, -1)$:
 - (a) Λ contains F_2 , or
 - (b) A pencil of Λ contains F_2 , and an element of the pencil is singular along F_2 .
- (4) $\rho_4 = (2, 2, -1, -1, -1, -1)$:
 - (a) Λ contains F_1 , and an element of Λ is singular along F_1 , or
 - (b) A pencil of Λ is singular along F_1 .
- (5) $\rho_5 = (5, -1, -1, -1, -1, -1)$:
 - (a) Λ contains F_0 , and a pencil of Λ is singular at F_0 .

Proof. In case (5), $(4, -2, -2)$ and $(-2, -2, -2)$ are all triples of initial ρ_5 -weights with non-positive sum. Any quadric of weight 4 contains F_0 and any quadric of weight -2 is singular at F_0 . So the net with initial ρ_5 -weights $(4, -2, -2)$ or $(-2, -2, -2)$ has a base point at F_0 and contains a pencil of quadrics singular at F_0 . The proofs of cases (1) – (4) are similar. \square

On the basis of this partial analysis, we may already conclude the important fact that a stable net has a pure two-dimensional intersection, and hence defines a connected surface with local complete intersection singularities.

Corollary 2.5. *If a net of quadrics in \mathbb{P}^5 is stable, then the corresponding intersection is connected and purely two-dimensional.*

Proof. The proof is basically same as the proof of Corollary 2.8. in [7] and so we omit details.

Connectedness follows from Fulton-Hansen connectedness theorem [8]. Suppose the intersection fails to be purely 2-dimensional. Then either a pencil of quadrics in the net contains a hyperplane, in which case the net is not ρ_1 -stable, or we may choose a basis $\{Q_1, Q_2, Q_3\}$ of the net such that $Y := Q_1 \cap Q_2$ is a quartic 3-fold and there is an irreducible component $Y' \subset Y$ of degree at most 3 which is contained in Q_3 . The net is not ρ_2 -stable (resp., not ρ_1 -stable) if the degree of Y' is 1 (resp., 2).

If $\deg Y' = 3$, then Y' is a rational normal scroll. If Y' is smooth, then the net is projectively equivalent to $(x_0x_3 - x_1x_2, x_0x_5 - x_1x_4, x_2x_5 - x_3x_4)$ and is not ρ_4 -stable. If Y' is singular, then it is either $S_{0,0,3}$ or $S_{0,1,2}$. If

$Y' = S_{0,0,3}$, then it is singular along a line. So we must have a pencil of quadrics singular along F_1 after the coordinate change. Such a net is not ρ_4 -stable. If Y' is a cone over $S_{1,2}$ and F_0 denotes the vertex of the cone, then we must have a pencil of quadrics singular at F_0 and the net contains F_0 . Such a net is not ρ_5 -stable. \square

Remark 2.6. *The Segre 3-fold $(x_0x_3 - x_1x_2, x_0x_5 - x_1x_4, x_2x_5 - x_3x_4)$ is strictly semistable. It is stabilized by a certain 1-PS acting diagonally with respect to the distinguished basis $\{x_0, x_1, \dots, x_5\}$. Indeed, it is stabilized by $(4, 2, 1, -1, -2, -4)$. By the Kempf-Morisson criterion (Proposition 2.4 in [1]), it therefore suffices to check that it is semistable with respect to 1-PS's acting diagonally with respect to this basis.*

Now, let $\lambda = (a, b, c, d, e, f)$ be a 1-PS with $a+b+c+d+e+f = 0$. Suppose it is unstable with respect to λ . Then $\omega_\lambda(x_0x_3 - x_1x_2) + \omega_\lambda(x_0x_5 - x_1x_4) + \omega_\lambda(x_2x_5 - x_3x_4) = \max\{a+d, b+c\} + \max\{a+f, b+e\} + \max\{c+f, d+e\} < 0$. In particular, $0 > (a+d) + (b+e) + (c+f) = a+b+c+d+e+f = 0$. It is a contradiction. Thus the Segre 3-fold is semistable. But it is not ρ_4 -stable and so is strictly semistable.

Lemma 2.7. *Suppose Λ is not ρ_i -stable for $i \in \{6, \dots, 13\}$ but is ρ_j -stable for $1 \leq j \leq i-1$. Let m_1, m_2, m_3 be the initial monomials of Λ with respect to ρ_i . Then $(\omega_{\rho_i}(m_1), \omega_{\rho_i}(m_2), \omega_{\rho_i}(m_3))$ must be one of the following triples:*

- (6) $\rho_6 = (2, 2, 2, -1, -1, -4)$:
• $(4, -2, -2)$
- (7) $\rho_7 = (7, 1, 1, 1, -5, -5)$:
• $(2, 2, -4)$
- (8) $\rho_8 = (4, 1, 1, -2, -2, -2)$:
• $(2, 2, -4)$
- (9) $\rho_9 = (3, 1, 1, -1, -1, -3)$:
• $(2, 0, -2)$
- (10) $\rho_{10} = (2, 1, 0, 0, -1, -2)$:
• $(2, 0, -2)$
- (11) $\rho_{11} = (5, 5, -1, -1, -1, -7)$:
• $(4, -2, -2)$
- (12) $\rho_{12} = (1, 1, 0, 0, -1, -1)$:
• $(0, 0, 0)$
- (13) $\rho_{13} = (5, 3, 1, -1, -3, -5)$:
• $(2, 0, -2)$

Proof. Consider $\rho_i = (a, b, c, d, e, f)$ for $6 \leq i \leq 13$ and suppose $\omega_1 \geq \omega_2 \geq \omega_3$ is the triple of ρ_i -initial weights of a ρ_i -nonstable net Λ . We can translate Lemma 2.3 into the following conditions:

- (C1) $\omega_3 \geq 2e$.
- (C2) $\omega_2 \geq 2d$ and $\omega_3 \geq d + f$.
- (C3) $\omega_1 \geq 2c$. Moreover, if $\omega_2 < 2c$, then $\omega_3 \geq c + f$.
- (C4) $\omega_2 \geq b + f$. Moreover, if $\omega_1 < 2b$, then $\omega_3 \geq b + f$.
- (C5) If $\omega_1 \neq 2a$, then $\omega_1 \geq a + e$ and $\omega_2 \geq a + f$.

Now for each ρ_i , we list all triples of ρ_i -initial weights with non-positive sum and satisfy (C1) – (C5). We will do only case (13), by far the most involved, and the remaining cases can be obtained more easily.

The set of possible $\rho_{13} = (5, 3, 1, -1, -3, -5)$ -weights of quadratic monomials is

$$\{10, 8, 6, 4, 2, 0, -2, -4, -6, -8, -10\}.$$

Suppose $\omega_1 \geq \omega_2 \geq \omega_3$ are initial ρ_{13} -weights of ρ_{13} -nonstable net Λ and Λ is ρ_i -stable for $1 \leq i \leq 12$. By (C2), $\omega_2 \geq -2$ and $\omega_3 \geq -6$. If $\omega_1 = 10$, then there is no triple with non-positive sum. Suppose $\omega_1 < 10$, then $\omega_1 \geq 2$ and $\omega_2 \geq 0$ by (C5). The triples with non-positive sum satisfying these conditions are

- $(6, 0, -6)$, which violates (C3);
- $(4, 2, -6)$, which violates (C4);
- $(4, 0, -6)$, which violates (C4);
- $(4, 0, -4)$, which violates (C4);
- $(2, 2, -4)$, which violates (C4);
- $(2, 2, -6)$, which violates (C4);
- $(2, 0, -4)$, which violates (C4);
- $(2, 0, -6)$, which violates (C4);
- $(2, 0, -2)$.

□

3. A NET OF QUADRICS AND THE ASSOCIATED DISCRIMINANT

To a net $\Lambda = (Q_1, Q_2, Q_3)$ of quadrics in \mathbb{P}^5 , we associate the sextic polynomial $\Delta(\Lambda) = \det(xQ_1 + yQ_2 + zQ_3)$ which is called the discriminant of Λ .

While the GIT stability of a pencil of quadrics can be read off the GIT stability of the associated discriminant locus (cf. [3]), the GIT analysis for nets of quadrics turns out to be more involved. In particular, as following example shows, there is no natural correspondence between $SL(6)$ -stability of a net and $SL(3)$ -stability of the associated discriminant curve.

Example 3.1. Let $\Lambda = (2x_0x_4 + 2x_1x_3, 2x_0x_5 + 2x_1x_4 + 2x_2x_3, 2x_1x_5 - 2x_2x_4)$. Then Λ is strictly semistable. It is stabilized by a certain 1-PS acting diagonally with respect to the distinguished basis $\{x_0, x_1, \dots, x_5\}$. Indeed, it is stabilized by $(3, 2, 1, -1, -2, -3)$. By the Kempf-Morisson criterion (Proposition 2.4 in [1]), it therefore suffices to check that it is semistable with respect to 1-PS's acting diagonally with respect to this basis. Suppose it is unstable with respect to a 1-PS $\lambda = (a, b, c, d, e, f)$ with $a+b+c+d+e+f = 0$. Then $\omega_\lambda(2x_0x_4 + 2x_1x_3) + \omega_\lambda(2x_0x_5 + 2x_1x_4 + 2x_2x_3) + \omega_\lambda(2x_1x_5 - 2x_2x_4) = \max\{a+e, b+d\} + \max\{a+f, b+e, c+d\} + \max\{b+f, c+e\} < 0$. In particular, $0 > (a+e) + (c+d) + (b+f) = a+b+c+d+e+f = 0$. It is a contradiction.

Thus Λ is semistable. But the discriminant sextic $\Delta(\Lambda)$ of Λ is $-y^6$, which is unstable under the natural $SL(3)$ -action on the space of plane sextic since $\omega_\chi(-y^6) = -6 < 0$ for $\chi = (2, -1, -1)$.

Proposition 3.2. If the discriminant of Λ is identically zero then Λ is unstable.

Proof. Since a net Λ of quadrics in \mathbb{P}^5 consists of singular quadrics, we have one of the followings (cf. Corollary 1 in [20]):

- (1) Quadrics in Λ have a common singular point; or
- (2) Restricted to a common hyperplane, the quadrics in Λ are singular along a line; or
- (3) Restricted to a common 3-dimensional linear space, the quadrics in Λ are singular along a plane.

In case (3), we can take coordinates x_0, x_1, \dots, x_5 such that the common 3-dimensional linear space T is $x_4 = x_5 = 0$ and the plane in T is defined by $x_3 = 0$. So the matrix with respect to any quadric in Λ has a form :

$$\begin{bmatrix} 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix}.$$

(Here 0 denote a zero entry and * that no restriction is imposed.)

Then Λ is $(3, 3, 3, -1, -4, -4)$ -unstable.

Similarly, in cases (1) and (2), we may assume the matrix with respect to any quadric in Λ has a form :

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix}, \text{ respectively.}$$

(Here 0 denote a zero entry and * that no restriction is imposed.)

And then Λ is ρ_5 -unstable and ρ_{11} -unstable, respectively. \square

If a net Λ defines a complete intersection surface S , we will use the net Λ and the defining surface S interchangeably. In particular, if a net is stable, then we will use the net and the defining surface interchangeably because a stable net defines a complete intersection surface by Corollary 2.5,.

In this section, we use the instability results of the previous section to discuss some connections between a net of quadrics and the associated discriminant sextic curve.

Theorem 3.3. *Suppose a net Λ of quadrics in \mathbb{P}^5 defines a complete intersection normal surface S . If the discriminant $\Delta(S)$ of S is stable then Λ is stable.*

Proof. Suppose a net Λ of quadrics in \mathbb{P}^5 defines a complete intersection normal surface S . Note that S is irreducible and reduced.

Now suppose Λ is not stable. Then by Theorem 2.1, there exist a basis of V such that Λ is not ρ_i -stable for some $1 \leq i \leq 13$. Then we will show that $\Delta(S)$ is not stable. The non-stability of Λ for $\{\rho\}_{1=1}^5$ uses Lemma 2.4 and for $\{\rho\}_{1=6}^{13}$ uses Lemma 2.7.

If Λ is not ρ_i -stable for $i \in \{1, 2, 3\}$, then S is not normal by Lemma 2.4 (1)–(3).

If Λ is not ρ_4 -stable, then either Λ contains F_1 and an element of Λ is singular along F_1 or a pencil of Λ is singular along F_1 with respect to some distinguished flag $F_0 \subset F_1 \subset F_2 \subset F_3 \subset F_4 \subset \mathbb{P}^5$ by Lemma 2.4 (4). If Λ contains F_1 , and an element of Λ is singular along F_1 then S is not normal. Hence, it suffices to consider the case when a pencil of Λ is singular along F_1 . That is $(Q_2, Q_3) \subset (x_2, x_3, x_4, x_5)^2$, where $\{x_0, \dots, x_5\}$ a basis of V which define the flag. Then $\Delta(S)$ is contained in a linear span of $\{x^i y^j z^k | i + j + k = 6 \text{ and } i \geq 2\}$. Thus $\Delta(S)$ is not stable (cf. 1.9 in [14]).

If Λ is not ρ_i -stable for $\{5, 7, 8, 9, 10, 13\}$, then $\Delta(S)$ is not stable by the same argument as the latter case of ρ_4 .

If Λ is not ρ_6 -stable, then by Lemma 2.7 (6), $(\omega_{\rho_6}(m_1), \omega_{\rho_6}(m_2), \omega_{\rho_6}(m_3)) = (4, -2, -2)$ for some basis $\{x_0, \dots, x_5\}$ of V . Then the equations of Q_i can be written as

$$\begin{aligned} Q_1 &: q_1(x_0, \dots, x_5) = 0, \\ Q_2 &: x_5 l_2(x_0, \dots, x_5) + q_2(x_3, x_4) = 0, \\ Q_3 &: x_5 l_3(x_1, \dots, x_5) + q_3(x_3, x_4) = 0. \end{aligned}$$

where q_i are quadratic polynomials and l_i are linear polynomials. Then S is singular along $C : x_3 = x_4 = x_5 = Q_1 = 0$ and so is not normal.

Similarly, if Λ is not ρ_i -stable for $i \in \{11, 12\}$, then S is singular along $L : x_2 = x_3 = x_4 = x_5 = 0$ for some basis $\{x_0, \dots, x_5\}$ of V and so is not normal. \square

Theorem 3.4. *Suppose a net Λ of quadrics in \mathbb{P}^5 defines a complete intersection surface S with the reduced discriminant $\Delta(S)$. If $\Delta(S)$ is stable then Λ is stable.*

Proof. Let a net Λ of quadrics in \mathbb{P}^5 define a complete intersection surface S with the reduced discriminant $\Delta(S)$.

Now suppose Λ is not stable. Then by Theorem 2.1 Λ is not ρ_i -stable for some $1 \leq i \leq 13$. Then we will show that $\Delta(S)$ is not stable. By the proof of Theorem 3.3, $\Delta(S)$ is not stable if Λ is not ρ_i -stable for some $i \in \{5, 7, 8, 9, 10, 13\}$ or a pencil of Λ is singular along F_1 with respect to some distinguished flag $F_0 \subset F_1 \subset F_2 \subset F_3 \subset F_4 \subset \mathbb{P}^5$, and so Λ is not ρ_4 -stable. Therefore we consider remaining cases.

If Λ is not $\rho_{12} = (1, 1, 0, 0, -1, -1)$ -stable, then by Lemma 2.7 (12), $(\omega_{\rho_{12}}(m_1), \omega_{\rho_{12}}(m_2), \omega_{\rho_{12}}(m_3)) = (0, 0, 0)$ for some basis $\{x_0, \dots, x_5\}$ of V .

Then the matrix A_i with respect to each Q_i has the form:

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & a_{04} & a_{05} \\ 0 & 0 & 0 & 0 & a_{14} & a_{15} \\ 0 & 0 & 2a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & a_{23} & 2a_{33} & a_{34} & a_{35} \\ a_{04} & a_{14} & a_{23} & a_{33} & 2a_{34} & a_{35} \\ a_{05} & a_{15} & a_{23} & a_{33} & a_{34} & 2a_{35} \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & b_{04} & b_{05} \\ 0 & 0 & 0 & 0 & b_{14} & b_{15} \\ 0 & 0 & 2b_{22} & b_{23} & b_{24} & b_{25} \\ 0 & 0 & b_{23} & 2b_{33} & b_{34} & b_{35} \\ b_{04} & b_{14} & b_{23} & b_{33} & 2b_{34} & b_{35} \\ b_{05} & b_{15} & b_{23} & b_{33} & b_{34} & 2b_{35} \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & c_{04} & c_{05} \\ 0 & 0 & 0 & 0 & c_{14} & c_{15} \\ 0 & 0 & 2c_{22} & c_{23} & c_{24} & c_{25} \\ 0 & 0 & c_{23} & 2c_{33} & c_{34} & c_{35} \\ c_{04} & c_{14} & c_{23} & c_{33} & 2c_{34} & c_{35} \\ c_{05} & c_{15} & c_{23} & c_{33} & c_{34} & 2c_{35} \end{bmatrix}$$

and so

$$\begin{aligned} \Delta(S) &= \det(xA_1 + yA_2 + zA_3) \\ &= -q_1^2 q_2 \end{aligned}$$

where q_1, q_2 are quadratic polynomials in x, y, z . Thus $\Delta(S)$ is not reduced.

If Λ is not ρ_3 -stable, then either Λ contains F_2 or a pencil of Λ contains F_2 , and an element of the pencil is singular along F_2 with respect to some distinguished flag $F_0 \subset F_1 \subset F_2 \subset F_3 \subset F_4 \subset \mathbb{P}^5$ by Lemma 2.4 (3). First consider the case when Λ contains F_2 . That is $(Q_1, Q_2, Q_3) \subset (x_3, x_4, x_5)$, where $\{x_0, \dots, x_5\}$ a basis of V which define the flag. Then the matrix A_i with respect to each Q_i has the form:

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & a_{03} & a_{04} & a_{05} \\ 0 & 0 & 0 & a_{13} & a_{14} & a_{15} \\ 0 & 0 & 0 & a_{23} & a_{24} & a_{25} \\ a_{03} & a_{13} & a_{23} & 2a_{33} & a_{34} & a_{35} \\ a_{04} & a_{14} & a_{23} & a_{33} & 2a_{34} & a_{35} \\ a_{05} & a_{15} & a_{23} & a_{33} & a_{34} & 2a_{35} \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & b_{04} & b_{05} \\ 0 & 0 & 0 & b_{13} & b_{14} & b_{15} \\ 0 & 0 & 0 & b_{23} & b_{24} & b_{25} \\ 0 & b_{13} & b_{23} & 2b_{33} & b_{34} & b_{35} \\ b_{04} & b_{14} & b_{23} & b_{33} & 2b_{34} & b_{35} \\ b_{05} & b_{15} & b_{23} & b_{33} & b_{34} & 2b_{35} \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & c_{05} \\ 0 & 0 & 0 & c_{13} & c_{14} & c_{15} \\ 0 & 0 & 0 & c_{23} & c_{24} & c_{25} \\ 0 & c_{13} & c_{23} & 2c_{33} & c_{34} & c_{35} \\ 0 & c_{14} & c_{23} & c_{33} & 2c_{34} & c_{35} \\ c_{05} & c_{15} & c_{23} & c_{33} & c_{34} & 2c_{35} \end{bmatrix}$$

and so

$$\begin{aligned}
\Delta(S) &= \det(xA_1 + yA_2 + zA_3) \\
&= \det \left(\begin{bmatrix} O & A \\ A^T & B \end{bmatrix} \right) \\
&= -(\det(A))^2
\end{aligned}$$

where

$$O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} a_{03}x & a_{04}x + b_{04}y & a_{05}x + b_{05}y + c_{05}z \\ a_{13}x + b_{13}y + c_{13}z & a_{14}x + b_{14}y + c_{14}z & a_{15}x + b_{15}y + c_{15}z \\ a_{23}x + b_{23}y + c_{23}z & a_{24}x + b_{24}y + c_{24}z & a_{25}x + b_{25}y + c_{25}z \end{bmatrix}.$$

Thus $\Delta(S)$ is not reduced.

Now consider the case when a pencil of Λ contains F_2 , and an element of the pencil is singular along F_2 . That is $(Q_2, Q_3) \subset (x_3, x_4, x_5)$ and $Q_3 \in (x_3, x_4, x_5)^2$, where $\{x_0, \dots, x_5\}$ a basis of V which define the flag. Then $\Delta(S)$ is contained in a linear span of $x^6, x^5y, x^5z, x^4y^2, x^4yz, x^4z^2, x^3y^3, x^3y^2z, x^3yz^2, x^3z^3, x^2y^4, x^2y^3z, x^2y^2z^2, xy^5, xy^4z, y^6$. Thus $\Delta(S)$ is not stable (cf. 1.9 in [14]).

Finally, if Λ is not ρ_i -stable for $i \in \{1, 2, 4, 6, 11\}$, then $\Delta(S)$ is not stable by the same argument as the latter case of ρ_3 . \square

Let S be a K3 surface of degree 8 in \mathbb{P}^5 , given by the complete intersection of three quadrics Q_1, Q_2, Q_3 . Associated to S is a K3 surface S' which is a double cover of \mathbb{P}^2 ramified over a sextic. Let Λ be the net of quadrics spanned by Q_0, Q_1, Q_2 , and $\psi : X \rightarrow \mathbb{P}^2$ the double cover of \mathbb{P}^2 branched along $\Delta(\Lambda)$. Then S' is also a K3 surface. And there is a dominant rational map $\phi : \mathcal{N}^{ss} // SL(6) \dashrightarrow \mathcal{C}_6^{ss} // SL(3)$ where \mathcal{N} the space of nets of quadrics in \mathbb{P}^5 and \mathcal{C}_6 the space of plane sextic curves. The degree of this map is equal to the number of non-effective theta characteristics on a general sextic curve which is $2^9(2^{10} + 1)$ (cf. [6] and [19]). Our original motivation in this paper is to compare the moduli space of algebraic K3 surfaces with degree 8 with the moduli space of K3 surfaces as a double cover of \mathbb{P}^2 , ramified over a sextic curve. It is well known that if S is nonsingular then $\Delta(S)$ is stable [4].

Theorem 3.5. *Suppose a net Λ of quadrics in \mathbb{P}^5 defines a complete intersection surface S . If S has simple singularities, then $\Delta(S)$ has simple singularities.*

We begin with some preparation for the proof of Theorem 3.5. We refer [21] for it. And we also refer [9] for the singularity theory.

We consider a net Λ of quadrics given by the vanishing of

$$F(\lambda, x) \equiv \sum_{i=1}^3 \sum_{j,k=0}^5 a_{jk}^i \lambda_i x_j x_k = x^T \left(\sum_{i=1}^3 \lambda_i A_i \right) x,$$

where the matrices $A_i = (a_{jk}^i)$ are symmetric. Define the *total variety*

$$V = \{(\lambda, x) : F(\lambda, x) = 0\},$$

the *variety of base points*

$$B = \{x : F(\lambda, x) = 0 \text{ for all } \lambda\},$$

and the *discriminant*

$$\Delta := \Delta(\Lambda) = \{\lambda : \det(\sum_{i=1}^3 \lambda_i A_i) = 0\}.$$

Write also for any λ , Q_λ for the quadric $\{x : F(\lambda, x) = 0\}$, and Q_i for the quadric $x^T A_i x = 0$.

For any variety X , we write $S(X)$ for the variety of its singular points: we first consider this as a point set. Thus $S(Q_\lambda)$ is the vertex of the quadric Q_λ .

Lemma 3.6. ([21], Lemma 1.1.) $(\lambda, x) \in S(V)$ if and only if $x \in S(Q_\lambda) \cap B$.

Lemma 3.7. ([21], Lemma 1.2.) $x \in S(B)$ if and only if there exists $\lambda \in \Delta$ with $x \in S(Q_\lambda)$.

Combining these lemmas, we see that the image of $S(V)$ under the projection on \mathbb{P}^5 defined by x is precisely $S(B)$. This projection often gives a bijection $S(V) \rightarrow S(B)$. Indeed given $x \in B$, we again consider the tangent planes $x^T A_i y = 0$. In general these are independent, spanning a 3-dimensional vector space. We have $x \in S(B)$ when they are dependent: call x *tame* if they span a 2-dimensional space. In this case there is a unique linear relation (up to scalar multiples), hence a unique $\lambda \in \mathbb{P}^2$ with $x \in S(Q_\lambda)$ and so $(\lambda, x) \in S(V)$. Therefore $S(V) \rightarrow S(B)$ is bijective if all points in $S(B)$ are tame.

Lemma 3.8. For a stable net Λ , all points of $S(B)$ are tame.

Proof. Suppose that $x \in S(B)$ is not tame. Then there are $\lambda_1 \neq \lambda_2$ such that x is on the vertex of $Q_{\lambda_1}, Q_{\lambda_2}$ and therefore Λ is not ρ_5 -stable by Lemma 2.4 (5). \square

Say that two isolated hypersurface singularities are *the same type* if either they are analytically equivalent or they can be reduced by analytic equivalence to hypersurfaces defined by two functions

$$f(z_1, \dots, z_k) = 0 \text{ in } \mathbb{C}^k$$

$$f(z_1, \dots, z_k) + \sum_{i=1}^t z_{k+i}^2 = 0 \text{ in } \mathbb{C}^{k+t}.$$

Here of course, $\sum_{i=1}^t z_{k+i}^2$ could be replaced by any nonsingular quadratic form.

Proposition 3.9. ([21], Proposition 1.3.) Let $(\lambda, x) \in S(V)$ where x is tame in $S(B)$. Then the two singular points have the same type.

So for proving Theorem 3.5 it is enough to show that if V has simple singularities, then Δ has simple singularities because that if S has simple singularities then Λ is stable ([18],[16],[17] and [11]).

Now take coordinates such that the point λ under investigation is at $\Lambda_1 = (1, 0, 0)$ and Q_1 (of corank $(k+1)$) has equation

$$\sum_{i=k+1}^5 x_i^2 = 0.$$

We can take affine coordinates by setting $\lambda_1 = 1$, but for now we retain all the x_i . We partition all the matrices into blocks, separating the first $(k+1)$ rows and columns from the remaining $(5-k)$; thus

$$M = A_1 + \lambda_2 A_2 + \lambda_3 A_3 = \begin{bmatrix} A & B^T \\ B & C \end{bmatrix}.$$

Here A, B and $C - I$ are homogeneous linear in the λ_2, λ_3 . So C is invertible at, and hence near Λ_1 . We use the identity

$$\begin{bmatrix} A & B^T \\ B & C \end{bmatrix} \begin{bmatrix} I & 0 \\ -C^{-1}B & I \end{bmatrix} = \begin{bmatrix} A - B^T C^{-1}B & B^T \\ 0 & C \end{bmatrix}$$

to compute the determinant of M . We find

$$\det M = \det C \det(A - B^T C^{-1}B).$$

so the equation of Δ can be written $0 = \det(A - U)$, where $U = B^T C^{-1}B$.

To study V , we partition the coordinate vector $x^T = (y^T, z^T)$ correspondingly. Then

$$F(\lambda, x) = x^T M x = y^T A y + 2y^T B^T z + z^T C z;$$

for fixed y (and small λ) this is a nonsingular quadratic in z , with centre $x = -C^{-1}B y$. Setting $z = z' - C^{-1}B y$, we have

$$F(\lambda, x) = z'^T C z' + y^T A y - y^T B^T C^{-1} B y = z'^T C z' + y^T (A - U) y,$$

and the singularity of this has the same type as that of $y^T (A - U) y$.

In all, the singularity of Δ is given by $0 = \det(A - U)$, those of V have the same type as those of $0 = y^T (A - U) y$.

Remark 3.10. While A is homogeneous linear in λ_2, λ_3 , all the terms in U have order ≥ 2 . Indeed, the terms of degree 2 in U are given by $B^T B$. The tangent cone of Δ at Λ_1 is given by the terms of lowest degree in $\det(A - U)$, i.e. by $\det(A)$.

We can now begin the proof of Theorem 3.5.

Proof of Theorem 3.5. Consider the projection $S(V) \rightarrow \Delta$. If Λ_1 is an isolated singularity of Δ , the corank $(k+1)$ of Q_1 must satisfy $k \leq 2$. The results for $k = 0$ are contained in Theorem 1.4. of [21].

Now let $k = 1$. The net cut on the vertex $S(Q_1)$ (a projective line) is spanned by two forms, so can be reduced to one of the normal forms

$$(1) \lambda x^2 + \mu y^2 \quad (2) 2\lambda xy + \mu x^2 \quad (3) 2\lambda xy \quad (4) \lambda x^2 \quad (5) 0,$$

with respective discriminants

$$(1) \lambda \mu \quad (2) -\lambda^2 \quad (3) -\lambda^2 \quad (4) 0 \quad (5) 0.$$

The corresponding singularities of V (or B) are the base points of this system. For case (5) we get the whole line as a non-isolated singularity: this we will not discuss further. In case (2),(4) we get one point Y ($x = 0$); in case (3) two points X ($y = 0$) and Y , and in case (1) no singularities.

Since the above discriminant is the tangent cone to Δ at Λ_1 , we have a singularity of type A_1 in case (1), a higher double point A_n in cases (2) and (3) (cf. Theorem 3.2. and Theorem 3.3 in [21]).

In case (4), we can normalize

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix};$$

we also write

$$U = \begin{bmatrix} u & v \\ v & w \end{bmatrix}.$$

Then Δ is given (locally) by

$$(u - \lambda)w = v^2$$

and we may consider V as given (near Y) by

$$(u - \lambda)x^2 + 2vx + w = 0$$

in affine coordinates ($y = 1$).

Here we must first look at the terms of degree 2 in w , giving a homogeneous quadratic q in λ, μ . If q has distinct factors then V has a singularity of corank 1 and Δ one belonging to the D -series. If q has a repeated root which is not λ , then Δ has a singularity of type D_m for some m . If q is a multiple of λ^2 , Δ still has a triple point (of higher type) and V a corank 2 singularity. But the singularity of V is a simple (actually D_m) only if the coefficient of μ^2 in v does not vanish. And in this case Δ has a singularity of type E_6 .

Finally, let $k = 2$. The vertex of the quadric Q_1 is a (projective) plane, and the net cuts a pencil of conics (defined by Q_2 and Q_3) in this plane. The corresponding points of $S(V)$ are the intersections of these conics: we wish these to be isolated. We may thus have a singular pencil of type (x^2, y^2) or one of the five types of nonsingular pencil.

The tangent cone to Δ at Λ_1 is given by the discriminant of the pencil. If this has three distinct factors, Λ_1 has type D_4 .

If the discriminant of the pencil has one repeated point and one other, Λ_1 has type D_n for some n . We can take the pencil in the form

$$\mu(x_1^2 + 2\alpha x_0 x_1) + \nu(2x_0 x_2).$$

i.e.

$$A = \begin{bmatrix} 0 & \alpha\mu & \nu \\ \alpha\mu & \mu & 0 \\ \nu & 0 & 0 \end{bmatrix}.$$

And we write

$$U = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}.$$

Then Δ is given (locally) by

$$\begin{aligned} 0 = & -cd^2 + 2bce - ae^2 - b^2f + adf + c^2\mu - 2\alpha ce\mu - af\mu + 2\alpha bf\mu \\ & + 2cd\nu - 2be\nu - \alpha^2 f\mu^2 - 2c\mu\nu + 2\alpha e\mu\nu - d\nu^2 + \mu\nu^2. \end{aligned}$$

Since the 3-jet of Δ is $\mu\nu^2$, Λ_1 has type D_n for some n .

If the discriminant of the pencil has a threefold point, we take the pencil as

$$\mu(x_1^2 - x_0x_2) + \nu(x_0^2 + 2\alpha x_0x_1).$$

i.e.

$$A = \begin{bmatrix} \nu & \alpha\nu & \frac{-\mu}{2} \\ \alpha\nu & \mu & 0 \\ \frac{-\mu}{2} & 0 & 0 \end{bmatrix}.$$

And we write

$$U = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}.$$

Then the corresponding singular points on V are $(x_0, x_1, x_2) = (0, 0, 1)$ and $(4\alpha^2, -2\alpha, 1)$ (if $\alpha \neq 0$).

Δ is given (locally) by

$$\begin{aligned} 0 = & -c^2d + 2bce - ae^2 - b^2f + adf + c^2\mu - cd\mu + be\mu - af\mu - 2\alpha ce\nu \\ & + e^2\nu + 2\alpha bf\nu - df\nu + c\mu^2 - \frac{1}{4}d\mu^2 - \alpha e\mu\nu + f\mu\nu - \alpha^2 f\nu^2 + \frac{1}{4}\mu^3. \end{aligned}$$

and we may consider V as given by

$$0 = (a - \nu)x_0^2 + 2(b - \alpha\nu)x_0x_1 + 2(c + \frac{\mu}{2})x_0 + (d - \mu)x_1^2 + 2ex_1 + f$$

in affine coordinates ($x_2 = 1$).

Let $F = (a - \nu)x_0^2 + 2(b - \alpha\nu)x_0x_1 + 2(c + \frac{\mu}{2})x_0 + (d - \mu)x_1^2 + 2ex_1 + f$ and write

$$\begin{aligned} c &= l\mu^2 + m\mu\nu + n\nu^2 + c_{\geq 3} \\ e &= i\mu^2 + j\mu\nu + k\nu^2 + e_{\geq 3} \\ f &= s\mu^2 + t\mu\nu + r\nu^2 + u\nu^3 + f_{\geq 3} \end{aligned}$$

Suppose $\alpha \neq 0$. We consider the following cases:

- I. $r \neq 0$;
- II. $r = 0$ and $t \neq 0$;
- III. $r = 0$ and $t = 0$.

⊙**Case I:** $r \neq 0$. Then the coefficient of ν^4 comes from $-\alpha^2 f \nu^2$ and is nonzero, so Λ_1 has E_6 type.

⊙**Case II:** $r = 0$ and $t \neq 0$. Then the coefficient of $\mu \nu^3$ comes from $-\alpha e \mu \nu - \alpha^2 f \nu^2$ and is $-\alpha k - \alpha^2 t$. So Λ_1 has E_7 type if $k \neq -\alpha t$. If $k = -\alpha t$, The coefficient of ν^5 comes from $-2\alpha c e \nu - e^2 \nu - \alpha^2 f \nu^2$ and is $-2\alpha n k + k^2 - \alpha^2 u$. So Λ_1 has E_8 type if $-2\alpha n k + k^2 - \alpha^2 u \neq 0$.

Now let $k = -\alpha t$ and $-2\alpha n k + k^2 - \alpha^2 u = 0$. Then Λ_1 is not simple. So we observe the singularity of V .

There exists an automorphism φ of $\mathbb{C}[[x_0, x_1, \mu, \nu]]$ such that $\varphi(F) = x_0^2 + x_1^2 + G(\mu, \nu)$ with $G \in \langle \mu, \nu \rangle^3$. Transform F by

$$T = \begin{cases} \begin{bmatrix} 0 & -2\sqrt{-s} & -t & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{\sqrt{s}} & -\frac{1}{\sqrt{-s}} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} & \text{if } s \neq 0 \\ \begin{bmatrix} 1 & -\sqrt{-1} & -t & 0 \\ 0 & 0 & 0 & 1 \\ 1 & \sqrt{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} & \text{if } s = 0 \end{cases}$$

Then we can assume that

$$F = x_0^2 + x_1^2 + F_{\geq 3}(\mu, \nu) + \sum_{i=0,1} x_i G_i(x_0, x_1, \mu, \nu)$$

with $G_i \in \langle x_0, x_1, \mu, \nu \rangle^2$. The coordinate change $x_i \mapsto x_i - \frac{1}{2} G_i$ for $i = 0, 1$, $\mu \mapsto \mu$ and $\nu \mapsto \nu$ yields

$$F = x_0^2 + x_1^2 + F_{\geq 3}(\mu, \nu) + F_{\geq 4}(\mu, \nu) + \sum_{i=0,1} x_i H_i(x_0, x_1, \mu, \nu)$$

with $H_i \in \langle x_0, x_1, \mu, \nu \rangle^3$. Continuing with H_i instead of G_i in the same manner, the last sum will be of arbitrary high order, hence 0 in limit.

The 3-jet of $G(\mu, \nu)$ is

$$\begin{aligned} & -t^2 \mu^3 + 2\alpha t \mu^2 \nu - 2n t \mu^3 + 2k \mu^2 \nu + u \mu^3 \\ & = (-t^2 - 2n t + u) \mu^3 + (2\alpha t + 2k) \mu^2 \nu \\ & = 0 \quad (\text{because } k = -\alpha t \text{ and } -2\alpha n k + k^2 - \alpha^2 u = 0). \end{aligned}$$

So the singular point of V is not simple.

⊙**Case III:** $r = 0$ and $t = 0$. Then the coefficient of $\mu \nu^3$ comes from $-\alpha e \mu \nu$ and is $-\alpha k$. So Λ_1 has E_7 type if $k \neq 0$. If $k = 0$, The coefficient of ν^5 comes from $-\alpha^2 f \nu^2$ and is $-\alpha^2 u$. So Λ_1 has E_8 type if $u \neq 0$.

Now let $k = 0$ and $u = 0$. Then Λ_1 is not simple. So we observe the singularity of V .

Same way as above, there exists an automorphism φ of $\mathbb{C}[[x_0, x_1, \mu, \nu]]$ such that $\varphi(F) = x_0^2 + x_1^2 + G(\mu, \nu)$ with $G \in \langle \mu, \nu \rangle^3$ and the 3-jet of $G(\mu, \nu)$ is zero. Thus the singular point of V is not simple.

Now suppose $\alpha = 0$. We consider the following cases:

- I. $r \neq 0$;
- II. $r = 0$ and $k \neq 0$;
- III. $r = 0$ and $k = 0$.

⊙**Case I:** $r \neq 0$. Then the coefficient of $\mu\nu^3$ comes from $f\mu\nu$ and is nonzero, so Λ_1 has E_7 type.

⊙**Case II:** $r = 0$ and $k \neq 0$. Then the coefficient of ν^5 comes from $e^2\nu$ and is k^2 . So Λ_1 has E_8 .

⊙**Case III:** $r = 0$ and $k = 0$. Then Λ_1 is not simple. So we observe the singularity of V . Again there exists an automorphism φ of $\mathbb{C}[[x_0, x_1, \mu, \nu]]$ such that $\varphi(F) = x_0^2 + x_1^2 + G(\mu, \nu)$ with $G \in \langle \mu, \nu \rangle^3$. We will show $G \in \langle \mu, \nu^2 \rangle^3$, then the singular point of V is not simple.

After transform F by T we can assume that

$$F = x_0^2 + x_1^2 + F_{\geq 3}(\mu, \nu) + \sum_{i=0,1} x_i G_i(x_0, x_1, \mu, \nu)$$

with $G_i \in \langle x_0, x_1, \mu, \nu \rangle^2$. The coordinate change $x_i \mapsto x_i - \frac{1}{2}G_i$ for $i = 0, 1$, $\mu \mapsto \mu$ and $\nu \mapsto \nu$ yields

$$\begin{aligned} F &= x_0^2 + x_1^2 + F_{\geq 3}(\mu, \nu) - x_0 G_0 - x_1 G_1 + \frac{1}{4}G_0^2 + \frac{1}{4}G_1^2 \\ &\quad + (x_0 - \frac{1}{2}G_0) \cdot G_0(x_0 - \frac{1}{2}G_0, x_1 - \frac{1}{2}G_1, \mu, \nu) \\ &\quad + (x_1 - \frac{1}{2}G_1) \cdot G_1(x_0 - \frac{1}{2}G_0, x_1 - \frac{1}{2}G_1, \mu, \nu) \end{aligned}$$

Let

$$G_0 := G_0(x_0, x_1, \mu, \nu) = g_1 + x_0 g_2 + x_1 g_3 + q + x_0^2 g_4 + x_0 x_1 g_5 + x_1^2 g_6,$$

$$G_1 := G_1(x_0, x_1, \mu, \nu) = h_1 + x_0 h_2 + x_1 h_3 + q' + x_0^2 h_4 + x_0 x_1 h_5 + x_1^2 h_6$$

with $g_1, h_1 \in \langle \mu, \nu \rangle^2$, $g_2, g_3, h_2, h_3 \in \langle \mu, \nu \rangle$, q, q' is quadratic in x_0, x_1 and $g_4, g_5, g_6, h_4, h_5, h_6 \in \langle x_0, x_1, \mu, \nu \rangle$.

Then

$$\begin{aligned} &(x_0 - \frac{1}{2}G_0) \cdot G_0(x_0 - \frac{1}{2}G_0, x_1 - \frac{1}{2}G_1, \mu, \nu) \\ &= x_0 G_0 - \frac{1}{2}G_0^2 + \frac{1}{4}G_0^2 g_2 + \frac{1}{4}G_0 G_1 g_3 + x_0 g' + x_1 g'' \end{aligned}$$

and

$$\begin{aligned} &(x_1 - \frac{1}{2}G_1) \cdot G_1(x_0 - \frac{1}{2}G_0, x_1 - \frac{1}{2}G_1, \mu, \nu) \\ &= x_1 G_1 - \frac{1}{2}G_1^2 + \frac{1}{4}G_0 G_1 h_2 + \frac{1}{4}G_1^2 h_3 + x_0 h' + x_1 h'' \end{aligned}$$

with $g', g'', h', h'' \in \langle x_0, x_1, \mu, \nu \rangle^3$. So

$$\begin{aligned} F &= x_0^2 + x_1^2 + F_{\geq 3} - \frac{1}{4}G_0^2 - \frac{1}{4}G_1^2 + \frac{1}{4}G_0^2 g_2 + \frac{1}{4}G_0 G_1 g_3 + \frac{1}{4}G_0 G_1 h_2 \\ &\quad + \frac{1}{4}G_1^2 h_3 + F_{\geq 6} + x_0 \tilde{G}_0 + x_1 \tilde{G}_1 \end{aligned}$$

with $F_{\geq 6} \in \langle x_0, x_1, \mu, \nu \rangle^6$ and $\tilde{G}_0, \tilde{G}_1 \in \langle x_0, x_1, \mu, \nu \rangle^3$.

Continuing this process, we can obtain

$$F = x_0^2 + x_1^2 + F_{\geq 3} - \frac{1}{4}g_1^2 - \frac{1}{4}h_1^2 + \frac{1}{4}g_1^2g_2 + \frac{1}{4}g_1h_1g_3 + \frac{1}{4}g_1h_1h_2 + \frac{1}{4}h_1^2h_3 + \tilde{F}_{\geq 6},$$

where $\tilde{F}_{\geq 6} \in \langle x_0, x_1, \mu, \nu \rangle^6$.

Thus the 3-jet comes from

$$F_{\geq 3},$$

ν^4 term and $\mu\nu^3$ term comes from

$$F_{\geq 3} - \frac{1}{4}g_1^2 - \frac{1}{4}h_1^2$$

and ν^5 term comes from

$$F_{\geq 3} - \frac{1}{4}g_1^2 - \frac{1}{4}h_1^2 + \frac{1}{4}g_1^2g_2 + \frac{1}{4}g_1h_1g_3 + \frac{1}{4}g_1h_1h_2 + \frac{1}{4}h_1^2h_3.$$

Now, recall that V is given by

$$F = (a - \nu)x_0^2 + 2b - x_0x_1 + 2(c + \frac{\mu}{2})x_0 + (d - \mu)x_1^2 + 2ex_1 + f$$

in affine coordinates ($x_2 = 1$). Transform F by T and consider 3-jet, ν^4 term, $\mu\nu^3$ term and ν^5 term.

If $s \neq 0$, by using

$$T = \begin{bmatrix} 0 & -2\sqrt{-s} & -t & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{\sqrt{s}} & -\frac{1}{\sqrt{-s}} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

the 3-jet of $F_{\geq 3}$ is $(-t^2 - 2nt + u)\mu^3$ and so $F_{\geq 3} \in \langle \mu, \nu^2 \rangle^3$.

Also we can obtain the followings:

$x_0\nu^2$ term of x_0g_1 is $-\frac{1}{\sqrt{s}}x_0\nu^2$ and so ν^2 term of g_1 is $-\frac{1}{\sqrt{s}}\nu^2$,

$x_1\nu^2$ term of x_1h_1 is $\frac{1}{\sqrt{-s}}x_1\nu^2$ and so ν^2 term of h_1 is $\frac{1}{\sqrt{-s}}\nu^2$,

$x_0\mu\nu$ term of x_0g_1 is $\frac{j}{\sqrt{s}}x_0\mu\nu$ and so $\mu\nu$ term of g_1 is $\frac{j}{\sqrt{s}}\mu\nu$,

$x_1\mu\nu$ term of x_1h_1 is $-\frac{j}{\sqrt{-s}}x_1\mu\nu$ and so $\mu\nu$ term of h_1 is $-\frac{j}{\sqrt{-s}}\mu\nu$,

$x_0^2\nu$ term of $x_0^2g_2$ is $\frac{i}{3}x_0^2\nu$ and so ν term of g_2 is $\frac{i}{3}\nu$,

$x_1^2\nu$ term of $x_1^2h_3$ is $-\frac{i}{3}x_1^2\nu$ and so ν term of h_3 is $-\frac{i}{3}\nu$,

$x_0x_1\nu$ term of $x_0x_1(g_3 + h_2)$ is $-\frac{2i}{3\sqrt{-1}}x_0x_1\nu$ and so ν term of $g_3 + h_2$ is $-\frac{2i}{3\sqrt{-1}}\nu$.

Thus ν^4 term, $\mu\nu^3$ term and ν^5 term are 0. In all, $G \in \langle \mu, \nu^2 \rangle^3$.

In case $s = 0$, by using

$$T = \begin{bmatrix} 1 & -\sqrt{-1} & -t & 0 \\ 0 & 0 & 0 & 1 \\ 1 & \sqrt{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

we can similarly show $G \in \langle \mu, \nu^2 \rangle^3$. □

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